CATEGORY THEORY
TOPIC III: RELATIONS

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1. Relations

Definition 1. Let A be a set. A *relation on* A is a subset $R \subset A \times A$. If R be a relation on A, we write aRb to mean $(a, b) \in R$.

Example 1. The concept of "less than or equal to" can be expressed as a relation. Let $\mathbb{R}$ be the set of all real numbers. Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$. Then R is a relation, which in effect describes what it means for x to be less than or equal to y by listing all instances of this phenomenon. Thus, in this case, xRy means $x \leq y$.

It should be noted that we may generalize the notion of relation to subsets of the cartesian product of any sets. This is appropriate and useful in some cases. For example, this is precisely the origin of the terminology regarding "relational databases" in computer science. When we use the word relation, however, we will restrict our attention to *binary relations* on a single set; hence the definition above.

Definition 2. Let $R \subset A \times A$ be a relation on a set A. We define its *domain* to be

$$\text{dom}(R) = \{a \mid (a, b) \in R \text{ for some } b\},$$

and its *range* to be

$$\text{rng}(R) = \{b \mid (a, b) \in R \text{ for some } a\}.$$

Example 2. Let $\mathbb{R}$ be the set of all real numbers. Let

$$R = \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}.$$

Then R is a relation on the set $\mathbb{R}$; we know that the graph of R is an ellipse. Note that $\text{dom}(R) = [-a, a]$ and $\text{rng}(R) = [-b, b]$.

Example 3. Suppose that A is the set of all inhabitants of some island. Let U be the subset of $A \times A$ given by

$$(a, b) \in U \Leftrightarrow a \text{ is the uncle of } b.$$

Let N be the subset of $A \times A$ given by

$$(a, b) \in N \Leftrightarrow a \text{ is the niece of } b.$$

Note that aNb does not imply bUa, nor does aUb imply aNb. However, if we had $S \subset A \times A$ given by

$$(a, b) \in S \Leftrightarrow a \text{ is the sibling of } b,$$

then $aSb \Leftrightarrow bSa$.

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Normally, relations are denoted by symbols other than letters. Thus we adopt the convention that the symbol $\bowtie$ will denote a generic relation. Keep in mind that if $\bowtie$ is a relation on a set A, this means $\bowtie$ is a subset of $A \times A$. However, it is also useful to regard $\bowtie$ as a "relational binary operator" which takes two elements of the set A and returns either TRUE or FALSE.

2. Properties of Relations

Definition 3. Let $\bowtie$ be a relation on a set A.
The relation is *reflexive* if $a \bowtie a$ for all $a \in A$.
The relation is *symmetric* if $a \bowtie b$ implies $b \bowtie a$.
The relation is *transitive* if $a \bowtie b$ and $b \bowtie c$ implies $a \bowtie c$.
The relation is *antisymmetric* if $a \bowtie b$ and $b \bowtie a$ implies $a = b$.
The relation is *definite* if $a \bowtie b$ or $b \bowtie a$ for all $a, b \in A$.

Example 4. The relation "is the same person as" is reflexive, symmetric, and transitive; so is the relation "is the same height as". The relation "is the parent of" has none of these properties (except antisymmetry; think about why). The relation "is the ancestor of" is transitive, and if we allow that one is one's own ancestor, it is also reflexive and antisymmetric.

Proposition 1. *Let $\bowtie$ be a relation on a set A.*
If $\bowtie$ is reflexive, then $\mathrm{dom}(\bowtie) = A$.

Proof. Let $a \in A$. Since $\bowtie$ is reflexive, $a \bowtie a$. This means that $(a, a) \in \bowtie$, so $a \in \mathrm{dom}(\bowtie)$. $\square$

Proposition 2. *Let $\bowtie$ be a relation on a set A.*
If $\bowtie$ is definite, then $\bowtie$ is reflexive.

Proof. Let $a \in A$, and set $b = a$. Since $\bowtie$ is definite, either $a \bowtie b$ or $b \bowtie a$. But $b = a$, so $a \bowtie a$. $\square$

Definition 4. Let $\bowtie$ be a relation on a set A. Let $B \subset A$. The *restriction* of $\bowtie$ to B is

$$\bowtie\restriction_B = \bowtie \cap (B \times B).$$

For $b, c \in B$, we still write $b \bowtie c$ to mean $(b, c) \in R \cap (B \times B)$.

Proposition 3. *Let $\bowtie$ be a relation on a set A, and let $B \subset A$. Then the restriction of $\bowtie$ to B is a relation on B. Moreover, if $\bowtie$ is reflexive, symmetric, transitive, antisymmetric, or definite on A, then it still has this property when restricted to B.*

The bulk of what we need to say on this topic describes a certain type of relation known as an equivalence relation. The notion of equivalence relation may be the most fundamental abstract concept in mathematics, and mastering it is a prerequisite for defining numbers and studying abstract algebra and topology, or anything dependent on these topics.

However, we will also have need for certain types of order relations, so we discuss those first.

3. Order Relations

Definition 5. Let $\leq$ be a relation on a set A. We say that $\leq$ is a *partial order* if for all $a, b, c \in A$ we have

(OR1) $a \leq a$ (reflexivity);
(OR2) if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry);
(OR3) if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

A partial order $\leq$ is called a *total order* if additionally it satisfies

(O4) either $a \leq b$ or $b \leq a$ for all $a, b \in A$ (definiteness).

We note that **(O1)** is included by **(O4)**, so it is unnecessary to address **(O1)** to show that a relation is a total order.

Proposition 4. *Let X be a set. The containment relation $\subset$ is a partial order on the power set $\mathcal{P}(X)$.*

Proof. Let A, B, and C be any sets. We have already seen that

(OR1) $A \subset A$;
(OR2) $A \subset B$ and $B \subset A$ implies $A = B$;
(OR3) $A \subset B$ and $B \subset C$ implies $A \subset C$.

Thus if $A, B, C \in \mathcal{P}(X)$, the above statements remain true, which says that $\subset$ is a partial order on $\mathcal{P}(X)$. $\square$

Example 5. The containment relation on the power set is not a total order relation. For example, if $X = \{1, 2, 3, 4, 5\}$, then the subsets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ are not related by inclusion.

Example 6. Familiar examples of totally ordered sets are the natural number $\mathbb{N}$, the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, and the real numbers $\mathbb{R}$. The complex numbers $\mathbb{C}$ have no total ordering which is compatible with their algebraic structure. We do, however, have a several partial orderings on $\mathbb{C}$ which arise from their algebraic structure (think about what these could be).

Example 7. Let $X = \mathbb{Z} \times \mathbb{Z}$, and let $\leq$ be the standard total order on $\mathbb{Z}$. Define a relation $\curlywedge$ on X by

$$(a, b) \curlywedge (c, d) \;\Leftrightarrow\; (a \leq c) \wedge (b \leq d).$$

Show that $\curlywedge$ is a partial order.

Solution. We wish to show that $\curlywedge$ is reflexive, antisymmetric, and transitive.

(OR1) Let $(a, b) \in X$. Then since $\leq$ is a total order, it is reflexive, so $a \leq a$ and $b \leq b$. Thus $(a, b) \curlywedge (a, b)$, and R is reflexive.

(OR2) Let $(a, b), (c, d) \in X$ such that $(a, b) R (c, d)$ and $(c, d) \curlywedge (a, b)$. Then $a \leq c$ and $c \leq a$. Since $\leq$ is antisymmetric, we have $a = c$. Similarly, $b = d$. Thus $(a, b) = (c, d)$, and R is antisymmetric.

(OR3) Let $(a, b), (c, d), (e, f) \in X$ and suppose that $(a, b) \curlywedge (c, d)$ and $(c, d) \curlywedge (e, f)$. Then $a \leq c$ and $c \leq e$. Since $\leq$ is transitive, we have $a \leq e$. Similarly, $b \leq f$. Thus $(a, b) \leq (e, f)$, and R is transitive. $\square$

Consider the graph the set $X = \mathbb{Z} \times \mathbb{Z}$, so that we may visualize the set X as a set of discrete points in the plane $\mathbb{R}^2$. If we graph the point (a, b), the set of points in X greater than (a, b) are those lying to the right and above the position of (a, b).

Definition 6. When the symbol $\leq$ is used for a partial order on a set X, the following symbols are assumed to have the given meanings:

- $a < b$ means $a \leq b$ and $a \neq b$;
- $a \geq b$ means $b \leq a$;
- $a > b$ means $b < a$;
- $a \nleq b$ means $\neg(a \leq b)$;
- $a \ngeq b$ means $\neg(a \geq b)$;
- $a \nless b$ means $\neg(a < b)$;
- $a \ngtr b$ means $\neg(a > b)$.

Definition 7. Let $\leq$ be a partial order on a set X, and let $m \in X$.
We say that m is a *maximal* element of X if $m \nleq x$ for every $x \in X$.
We say that m is a *minimal* element of X if $x \nleq m$ for every $x \in X$.

Example 8. Let $X = \mathcal{P}(\mathbb{N}) \smallsetminus \varnothing$. Then X is partially ordered by containment, and all the singleton sets are minimal elements of X.

Definition 8. Let $\leq$ be a partial order on a set X, and let $m \in X$.
We say that m is a *maximum* element of X if $x \leq m$ for every $x \in X$.
We say that m is a *minimum* element of X if $m \leq x$ for every $x \in X$.
If m is a maximum or a minimum, then we say that m is an *extremum*.

It is clear that maximum elements and maximal, and that minimum elements are minimal. It is also clear that extreme elements are unique, which we now prove.

Proposition 5. *Let $\leq$ be a partial order on a set X.*
If X has a maximum, it is unique, and is denoted by $\max X$.
If X has a minimum, it is unique, and is denoted by $\min X$.

Proof. Let $m_1, m_2 \in X$. Suppose m_1 and m_2 are maxima for X. Then $x \leq m_1$ and $x \leq m_2$ for every $x \in X$. Since m_1 and m_2 are both in X, this implies that $m_2 \leq m_1$ and $m_1 \leq m_2$. Thus, by antisymmetry, $m_1 = m_2$. The demonstration if m_1 and m_2 are minima is analogous. $\square$

4. Equivalence Relations

Let A be a set and let

$$D = \{(a,b) \in A \times A \mid a = b\}.$$

The set D is called the *diagonal* of $A \times A$; it is formally the relation of equality. The three key attributes of this relation are that it is reflexive, symmetric, and transitive. The notion of equivalence relation generalizes this idea.

Definition 9. Let A be a set and let $\equiv$ be a relation on A. We say that $\equiv$ is an *equivalence relation* if it is reflexive, symmetric, and transitive:
(EQ1) $a \equiv a$ (reflexivity);
(EQ2) $a \equiv b$ if and only if $b \equiv a$ (symmetry);
(EQ3) if $a \equiv b$ and $b \equiv c$, then $a \equiv c$ (transitivity).

Example 9. Let A be the set of all animals in the world. Define a relation $\sim$ by

$$\sim = \{(a,b) \in A \times A \mid a \text{ and } b \text{ are of the same species } \}.$$

It is more traditional to write this as

$$a \sim b \Leftrightarrow a \text{ and } b \text{ are of the same species.}$$

Then $\sim$ is an equivalence relation on the set A. For certainly if an animal a is a pig, then it is a pig (reflexivity); if a and b are both pigs, then b and a are both pigs (symmetry); and if a and b are both pigs, and b and c are both pigs, then a and c are both pigs (transitivity).

Example 10. Let $X = \mathbb{N} \times \mathbb{N}$. Define a relation on X by

$$(a,b) \equiv (c,d) \Leftrightarrow a + d = b + c.$$

This is an equivalence relation.

Example 11. Let $\mathbb{Z}^\bullet = \mathbb{Z} \smallsetminus \{0\}$ be the set of nonzero integers. Let $X = \mathbb{Z} \times \mathbb{Z}^\bullet$. Define a relation on X by

$$(a,b) \equiv (c,d) \Leftrightarrow ad = bc.$$

Show that this is an equivalence relation.

Solution. We wish to show that $\equiv$ is reflexive, symmetric, and transitive.
(EQ1) Let $(a,b) \in X$. Then $ab = ba$ by commutativity of multiplication. This says that $(a,b) \equiv (a,b)$, so $\equiv$ is reflexive.
(EQ2) Let $(a,b),(c,d) \in X$. Then

$$(a,b) \equiv (c,d) \Leftrightarrow ad = bc \Leftrightarrow cb = da \Leftrightarrow (c,d) \equiv (a,b),$$

so $\equiv$ is symmetric.
(EQ3) Let $(a,b),(c,d),(e,f) \in X$. Suppose that $(a,b) \equiv (c,d)$ and $(c,d) \equiv (e,f)$. Then $ad = bc$ and $ce = df$. Multiply the first equation by e and the second by b and apply commutativity of multiplication in the integers to obtain $ade = bce$ and $bce = bdf$. Then by transitivity of equality, we have $ade = bdf$. By cancellation, we have $ae = bf$. Thus $(a,b) \equiv (e,f)$, and $\equiv$ is transitive. $\square$

5. Equivalence Classes

Equivalence relations are particularly important, because they group the elements of a set into blocks such that the members of one of the blocks, although not exactly equal, are similar in some sense in which one may be interested. More precisely, equivalence relations induce partitions on sets, and the blocks are called called equivalence classes.

Definition 10. Let $\equiv$ be an equivalence relation on a set A. We say that two element $a, b \in A$ are *equivalent* if $a \equiv b$. Since $\equiv$ is symmetric, this is the case if and only if $b \equiv a$. The *equivalence class* of a, denoted $[a]_\equiv$, is the set of all elements of A which are equivalent to a:

$$[a]_\equiv = \{b \in A \mid a \equiv b\}.$$

If the equivalence relation is understood, is is more convenient to simply write $[a]$ for the equivalence class, or more commonly, $\overline{a}$. We will use whichever of these notations is most convenient in a given circumstance.

Example 12. Suppose A is the set of all animals in the world, and $\equiv$ is the relation of being in the same species. Let p be a pig. Then $\overline{p}$ is the set of all pigs in the world. One can see that if q is also a pig, then $\overline{q}$ is also the set of all pigs in the world, so $\overline{p} = \overline{q}$. Also it is clear that if a is an anteater, then $\overline{p} \cap \overline{a} = \varnothing$. Note there is exactly one equivalence class $\overline{x}$ for each species of animal on earth such that x is an animal of that species. We now proceed to formalize these assertions.

Proposition 6. *Let A be a set and let $\equiv$ be an equivalence relation on A. For $a \in A$, let $[a]$ denote the equivalence class of a. Then the following conditions are equivalent:*

(i) $a \equiv b$;
(ii) $[a] = [b]$;
(iii) $b \in [a]$.

Proof. To prove a statement of this kind, we need to show that **(i)** is logically equivalent to **(ii)**, that **(ii)** is logically equivalent to **(iii)**, and that **(iii)** is logically equivalent to **(i)**. It suffices to show that **(i)** implies **(ii)**, that **(ii)** implies **(iii)**, and that **(iii)** implies **(i)**.

(i) $\Rightarrow$ **(ii)** Suppose that $a \equiv b$. By symmetry of $\equiv$, we know that $b \equiv a$. We wish to show that $[a] = [b]$. We show containment both ways.

Let $c \in [a]$. Then $a \equiv c$ by definition of $[a]$. Thus $b \equiv c$ by transitivity of $\equiv$, because $b \equiv a$ and $a \equiv c$. Thus $c \in [b]$ by definition of $[b]$. This shows that $[a] \subset [b]$.

Simply by reversing the roles of a and b is the above argument, we see that $[b] \subset [a]$. Therefore $[a] = [b]$.

(ii) $\Rightarrow$ **(iii)** Suppose that $[a] = [b]$. We wish to show that $b \in [a]$. Now by reflexivity, $b \equiv b$. Thus $b \in [b]$. Since $[a]$ is the same set as $[b]$, we must have $b \in [a]$.

(iii) $\Rightarrow$ **(i)** Suppose that $b \in [a]$. We wish to show that $a \equiv b$. But this follows by the definition of $[a]$. $\square$

6. Partitions induced by Equivalence Relations

Proposition 7. *Let A be a set and let $\equiv$ be an equivalence relation on A. Then the collection of equivalence classes*

$$\mathcal{C} = \{[a] \in \mathcal{P}(A) \mid a \in A\}$$

forms a partition of A.

Proof. We wish to show that the equivalence classes are nonempty, pairwise disjoint, and cover A. It is clear that they are nonempty and cover A, since for any $a \in A$, we have $a \in [a]$.

Let $a, b \in A$ so that $[a], [b] \in \mathcal{C}$ are arbitrary equivalence classes. Suppose that their intersection is nonempty, say $c \in [a] \cap [b]$. Then $[c] = [a]$ and $[c] = [b]$; thus $[a] = [b]$. This tells us that the only way two equivalence classes can have a nonempty intersection is if they are the same class. Thus distinct equivalence classes are disjoint. This was our condition to call the sets in a collection of subsets pairwise disjoint. $\square$

The collection of equivalence classes referred to above is called the *partition induced by the equivalence relation.*

Proposition 8. *Let A be a set and let $\mathcal{C}$ be a partition of A. Define a relation $\sim$ on A by*

$$\sim = \{(a, b) \in A \times A \mid a \in [b]\}.$$

Then $\sim$ is an equivalence relation.

Proof. We wish to show that $\sim$ is reflexive, symmetric, and transitive.

Since $\mathcal{C}$ is a partition, every element of $a \in A$ is in exactly one member of $\mathcal{C}$. Let us denote this member by $[a]$. We first note that for $a, b \in A$, $a \in [b]$ if and only if $[a] = [b]$. To see this, suppose that $a \in [b]$. Then $[b]$ is the unique member of the partition $\mathcal{C}$ which contains a. Since we are calling this member $[a]$, we have $[a] = [b]$. On the other hand, if $[a] = [b]$, we know that $a \in [a]$, so $a \in [b]$.

We have $a \in [a]$, so $(a, a) \in \sim$. Thus $\sim$ is reflexive.

Suppose $a \sim b$. We wish to show that $b \sim a$. Now $a \sim b$ means that $a \in [b]$, so $[a] = [b]$. Thus $a \in [b]$; therefore $b \sim a$. Reversing the roles of a and b shows that $b \sim a \Rightarrow a \sim b$. Thus $a \sim b \Leftrightarrow b \sim a$, and $\sim$ is symmetric.

Suppose that $a \sim b$ and $b \sim c$. We wish to show that $a \sim c$. Rephrased, we wish to show if $a \in [b]$ and $b \in [c]$, then $a \in [c]$. But $a \in [b]$ implies that $[a] = [b]$, and $b \in [c]$ implies that $[b] = [c]$; thus $[a] = [c]$, so $a \in [c]$, and $a \sim c$. Thus $\sim$ is transitive. $\square$

The relation defined above is called the *equivalence relation induced by the partition.* The above two propositions say that the concepts of partition and equivalence relation correspond to each other in a natural way. A partition is an equivalence relation by considering its blocks as equivalence classes, and an equivalence relation partitions the set into blocks which are equivalence classes.

7. Partitions induced by Functions

We now show that if $f : A \to B$ is a function, then f induces an equivalence relation on the domain A.

Proposition 9. *Let $f : A \to B$ be a function. Define a relation $\equiv$ on A by*

$$a_1 \equiv a_2 \Leftrightarrow f(a_1) = f(a_2).$$

Then $\equiv$ is an equivalence relation.

Proof. We wish to show that $\equiv$ is reflexive, symmetric, and transitive.

It is reflexive because $f(a) = f(a)$. It is symmetric because $f(a_1) = f(a_2) \Leftrightarrow f(a_2) = f(a_1)$. It is transitive because $f(a_1) = f(a_2)$ and $f(a_2) = f(a_3)$ implies that $f(a_1) = f(a_3)$. $\square$

The relation defined above is called the *equivalence relation induced by the function*, and the associated partition, naturally enough, is called the *partition induced by the function*. The blocks of this partition are nothing but the preimages of points in B under the map A. The equivalence relation induced by a function is sometimes called a *kernel equivalence*. The equivalence class of a under such an equivalence is sometimes denoted $\overline{a}$ instead of $[a]$. The set of equivalence classes may be denoted $\overline{A}$.

Example 13. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin x$. Then f induces an equivalence relation on $\mathbb{R}$ which is given by

$$x_1 \equiv x_2 \Leftrightarrow x_2 - x_1 = k\pi \text{ for some } k \in \mathbb{Z}.$$

The blocks of the corresponding partition are the equivalence classes of this equivalence relation. Such a block consists of points scattered on the real line at a distance of π from each other. The set of all such blocks covers the real line.

Example 14. Let A be the set of animals on earth and let S be the set of species. Define a function $f : A \to S$ by sending an animal to the species of which it is a member. Then the partition of A induced by f is the collection of subsets of A consisting of blocks such that all the animals in one block are of the same species, and any two animals of the same species are in the same block.

8. Functions defined on Partitions

Let A be a set and let $\mathcal{A}$ be a partition of A. For a given $a \in A$, let $[a]$ denote the block in $\mathcal{A}$ which contains a. We say that *a represents* the block $[a]$, or that a is a *choice of representative*. Suppose B is another set and we wish to define a function $\alpha : \mathcal{A} \to B$, and we do so by saying where each block $[a] \in \mathcal{A}$ should be sent in B. Perhaps we use some formula or algorithm which depends on the choice of representative $a_1 \in [a]$. Then we better be certain that, if a_2 is another element representing $[a]$, then the algorithm gives the same value for a_2 as it did for a_1.

Example 15. Let $X = \mathbb{R} \smallsetminus \{0\}$ be the set of nonzero real numbers. Let $Y = \{x \in X \mid x > 0\}$ be the set of positive real numbers and let $Z = X \smallsetminus Y$ be the set of negative real numbers. Then $\mathcal{X} = \{Y, Z\}$ is a partition of X.

If we attempt to define a function $f : \mathcal{X} \to \mathbb{Z}$ by $[x] \mapsto x^2$, this doesn't make sense, since $[1] = [2]$, but $f([1]) = 1$ and $f([2]) = 4$.

However, if we attempt to define a function $g : \mathcal{X} \to \mathbb{Z}$ by $[x] \mapsto \frac{x}{|x|}$, this function does make sense, since the entire block of positive numbers is sent to 1 and the entire block of negative number is sent to -1.

Let A be a set and let $\mathcal{A}$ be a partition of A. Let $g : A \to B$ be a function. Suppose we define a function $f : \mathcal{A} \to B$ by specifying $f([a]) = g(a) \in B$. If $g(a_1) = g(a_2)$ whenever $[a_1] = [a_2]$, we say the function is *well-defined*.

Example 16. Let V be the set of vertebrate animals in the world and let $\mathcal{V}$ be the set of equivalence classes of vertebrates of the same species.

Let $T = \{\text{fish}, \text{amph}, \text{rept}, \text{bird}, \text{mamm}\}$ be the set of types of vertebrates. Attempt to define $f : \mathcal{A} \to B$ by

$$f([v]) = \begin{cases} \text{fish} & \text{if } v \text{ is a fish;} \\ \text{amph} & \text{if } v \text{ is an amphibian;} \\ \text{rept} & \text{if } v \text{ is a reptile;} \\ \text{bird} & \text{if } v \text{ is a bird;} \\ \text{mamm} & \text{if } v \text{ is a mammal.} \end{cases}$$

Then f is well-defined, since all the vertebrates of the same species are of the same type.

However, if we attempt to define $g : \mathcal{A} \to \mathbb{R}$ by

$$g([v]) = \text{ the mass of } v \text{ in grams },$$

then g is not well-defined, because not every vertebrate of the same species has the same mass.

9. Canonical Functions

Let $\overline{A}$ be a partition of a set A, and for $a \in A$ let $\overline{a}$ denote the block containing a. Then there is a *canonical function*

$$\beta : A \to \overline{A}$$

given by $f(a) = \overline{a}$. Each element simply is sent to the block containing it. That is, each element is sent to its equivalence class in the equivalence relation corresponding to the partition. The function β is surjective, since every block contains an element (we made it part of our definition of partition that its members are nonempty).

Theorem 1. *Let $\phi : A \to B$ be a function. Let $\overline{A}$ be the set of equivalence classes of A induced by f. Let $\beta : A \to \overline{A}$ be the canonical function given by $a \mapsto \overline{a}$. Then there exists a unique injective function*

$$\overline{\phi} : \overline{A} \to B$$

such that $\phi = \overline{\phi} \circ \beta$. If ϕ is surjective, then $\overline{\phi}$ is bijective.

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\beta"] & & B \\ & \overline{A} \arrow[ur, "\overline{\phi}"] & \end{tikzcd}$$

Proof. Define $\overline{\phi}$ by $\overline{\phi}(\overline{a}) = \phi(a)$. We must show that this is well defined and injective, that $\phi = \overline{\phi} \circ \beta$, and that any other function $\psi : \overline{A} \to B$ such that $\phi = \psi \circ \beta$ is equal to $\overline{\phi}$.

Note that $\overline{\phi}$ is defined via a choice of representative for a given block in $\overline{A}$. To show that $\overline{\phi}$ is well-defined, we must show that the definition of $\overline{\phi}$ is independent of the choice of representative. Thus let $a_1, a_2 \in A$ such that $\overline{a_1} = \overline{a_2}$. Thus a_1 and a_2 are inverse images of the same point in B under the map ϕ. That is, $\phi(a_1) = \phi(a_2)$. Therefore $\overline{\phi}(\overline{a_1}) = \phi(a_1) = \phi(a_2) = \overline{\phi}(\overline{a_2})$, and $\overline{\phi}$ is well-defined.

To see that $\overline{\phi}$ is injective, let $\overline{a_1}, \overline{a_2} \in \overline{A}$ such that $\overline{\phi}(\overline{a_1}) = \overline{\phi}(\overline{a_2})$. Then $\phi(a_1) = \phi(a_2)$. By definition of kernel equivalence, $\overline{a_1} = \overline{a_2}$, so $\overline{\phi}$ is injective.

To see that $\phi = \overline{\phi} \circ \beta$, note that for $a \in A$, $\phi(a) = \overline{\phi}(\overline{a}) = \overline{\phi}(\beta(a))$. Thus this holds essentially by definition of $\overline{\phi}$ and of β.

Suppose that $\psi : \overline{A} \to B$ is another function such that $\phi = \psi \circ \beta$. Then $\psi(\overline{a}) = \phi(a) = \overline{\phi}(\overline{a})$, so $\overline{\phi} = \psi$ since it acts the same way on every element of its domain. Thus $\overline{a}$ is the unique function with this property. □

Example 17. Let A be the set of animals on earth and let S be the set of species. Let $\phi : A \to S$ be given by sending an animal to its species. Let $\overline{A}$ be the partition of A into subsets of A which contain all of the animals of a given species. Then $\overline{A}$ is the partition of A induced by ϕ. Let $\beta : A \to \overline{A}$ be the canonical function which sends an animal to the block which contains it. One can easily see that such blocks naturally correspond to the set of species. The bijective function $\overline{\phi}$, whose existence is guaranteed by the above theorem, sends each block to the species to which the animals in the block belong.

10. Exercises

Exercise 1. Let A and B be sets and let $\leq$ be a total order on B. Let $f : A \to B$ be a function and define a relation $\preccurlyeq$ on A by

$$a_1 \preccurlyeq a_2 \Leftrightarrow f(a_1) \leq f(a_2).$$

(a) Show that if f is injective, $\preccurlyeq$ is a total order on A.
(b) Give an example where f is not injective and $\preccurlyeq$ is not a partial order on A.

Exercise 2. Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$. Define a relation $\preccurlyeq$ on $\mathcal{C}$ by

$$A \preccurlyeq B \Leftrightarrow \exists \text{ injective } f : A \to B.$$

Is $\preccurlyeq$ a partial order on $\mathcal{C}$?

Exercise 3. Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$. Define a relation $\equiv$ on $\mathcal{C}$ by

$$A \equiv B \Leftrightarrow \exists \text{ bijective } f : A \to B.$$

Show that $\equiv$ is an equivalence relation.

Definition 11. A *circle in the cartesian plane* is a subset of $\mathbb{R}^2$ which is the set of all points equidistant from a given point, called its *center*; the common distance is called the *radius* of the circle. If $C \subset \mathbb{R}^2$ is a circle and $A \subset \mathbb{R}^2$, we say that A is *inside* C if for each $a \in A$, the distance from a to the center of C is less than or equal to the radius of the circle.

Exercise 4. Let $\mathcal{C} \subset \mathcal{P}(\mathbb{R}^2)$ be the collection of all circles in the cartesian plane. Define a relation $\preccurlyeq$ on $\mathcal{C}$ by

$$C_1 \preccurlyeq C_2 \Leftrightarrow C_1 \text{ is inside } C_2.$$

Is $\preccurlyeq$ a partial order on $\mathcal{C}$?

Exercise 5. Let $\mathcal{C} \subset \mathcal{P}(\mathbb{R}^2)$ be the collection of all circles in the cartesian plane. Define a relation $\preccurlyeq$ on $\mathcal{C}$ by

$$C_1 \preccurlyeq C_2 \Leftrightarrow \text{ the center of } C_1 \text{ is inside } C_2.$$

Is $\preccurlyeq$ a partial order on $\mathcal{C}$?

Exercise 6. Let $\mathcal{C} \subset \mathcal{P}(\mathbb{R}^2)$ be the collection of all circles in the cartesian plane. Define a relation $\equiv$ on $\mathcal{C}$ by

$$C_1 \equiv C_2 \Leftrightarrow C_1 \text{ and } C_2 \text{ have the same center }.$$

Is $\equiv$ an equivalence relation?

Exercise 7. Define a function $|\cdot| : \mathbb{R}^2 \to \mathbb{R}$ by

$$|(x, y)| = \sqrt{x^2 + y^2}.$$

Let $\mathcal{C}$ be the partition of $\mathbb{R}^2$ induced by this function.
Describe the members of $\mathcal{C}$.

Exercise 8. Let $X = \{1, 2, 3\}$. Define a function $f : \mathcal{P}(X) \setminus \{\varnothing\} \to X$ by

$$f(A) = \text{ the smallest member of A.}$$

Compute the partition of $\mathcal{P}(X)$ induced by the function f.

Exercise 9. Let $X = \mathbb{N} \times \mathbb{N}$. Define a relation $\equiv$ on X by

$$(a, b) \equiv (c, d) \Leftrightarrow a + d = b + c.$$

(a) Show that this is an equivalence relation.
(b) Describe the equivalence classes.
(c) Let $\mathcal{C}$ be the set of equivalence classes. Denote the equivalence class of (a, b) by $[a, b]$. Determine which of the following functions $f : \mathcal{C} \to \mathbb{R}$ are well defined:

- $f([a, b]) = a^2 + b^2$;
- $f([a, b]) = a^2 - 2ab + b^2$;
- $f([a, b]) = \frac{a}{b}$;
- $f([a, b]) = \sin(a - b)$.

Exercise 10. Let $a, n \in \mathbb{Z}$. We say that n *divides* a, and write $n \mid a$, if there exists $k \in \mathbb{Z}$ such that $kn = a$.

Define a relation $\equiv$ on $\mathbb{Z}$ by

$$a \equiv b \Leftrightarrow 6 \mid (a - b).$$

(a) Show that $\equiv$ is an equivalence relation.
(b) Describe the equivalence classes.
(c) Count the equivalence classes.
(d) Let $\mathcal{C}$ be the set of equivalence classes. Denote the equivalence class of a by $[a]$. Determine which of the following functions $f : \mathcal{C} \to \mathbb{Z}$ are well defined:

- $f([a]) = 3a$;
- $f([a]) = 3r$, where r is the remainder when a is divided by 6;
- $f([a]) = x$, where x is the remainder when $3a$ is divided by 6;
- $f([a]) = x$, where x is the remainder when a is divided by 3;
- $f([a]) = x$, where x is the remainder when a is divided by 5.

Exercise 11. Let X be a set and let $\mathcal{C} = \{C_1, \dots, C_m\}$ and $\mathcal{D} = \{D_1, \dots, D_n\}$ be partitions of X. Define

$$\mathcal{E} = \{C_i \cap D_j \mid C_i \in \mathcal{C}, D_j \in \mathcal{D}\}.$$

(a) Show that $\mathcal{E}$ is a partition of X.
(b) Describe the equivalence relation induced by $\mathcal{E}$ in terms of the equivalence relations induced by $\mathcal{C}$ and $\mathcal{D}$.

Exercise 12. Let X and Y be sets. Let $\sim$ be an equivalence relation on X and let $\approx$ be an equivalence relation on Y. Let $[X]$ and $[Y]$ denote the respective sets of equivalence classes. Show that there is an induced equivalence relation $\equiv$ on $X \times Y$. Denote the set of equivalence classes by $[X \times Y]$, and for $(x, y) \in X \times Y$, denote its equivalence class by $[x, y]$. Define a function

$$\phi : [X \times Y] \to [X] \times [Y]$$

by $[x, y] \mapsto ([x], [y])$. Show that ϕ is well-defined and bijective.

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